

Beltrami Differential Operators Defined in Metric Space

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Abstract

We present some properties of the first and second order Beltrami differential operators in metric space.

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1 Introduction

In the three dimensional space, in the case of orthogonal systems (x, y, z) , of elementary Geometry we make use of the differential operator:

$$\Delta_2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \quad (1)$$

We call this differential operator of function Φ , Laplacian operator, or Beltrami differential parameter of second kind.

More general in the case of a n dimensional metric space of curvilinear coordinates u^i and metric tensor

$$g_{ik} = g_{ik}(u^l) = g_{ik} \quad (2)$$

Where $g^{ij} = (g_{ij})^t = (g_{ij})^{(-1)}$

Definition 1.

$$\Delta_1(\Phi, \Psi) := \sum_{i,j=1}^N (g_{ij})^t (\partial_i \Phi \partial_j \Psi + \partial_j \Phi \partial_i \Psi) = 2 \sum_{i,j=1}^N (g_{ij})^t \partial_i \Phi \partial_j \Psi \quad (3)$$

The Beltrami operator of the second kind is

Definition 2.

$$\Delta_2 \Phi = \sum_{i,j=1}^N (g_{ij})^t \left(\frac{\partial^2 \Phi}{\partial u_i \partial u_j} - \Gamma_{ij}^k \partial_k \Phi \right) \quad (4)$$

Where the $\Gamma_{ki}^j = \Gamma_{ik}^j$ are called Christoffel symbols and related with g_{ij} from the relations

$$\partial_l g_{ik} = \sum_{n=1}^N g_{nl} \Gamma_{ki}^n + \sum_{n=1}^N g_{kn} \Gamma_{il}^n \quad (5)$$

and

$$\Gamma_{kl}^i = \frac{1}{2} \sum_{n=1}^N (g_{in})^t (\partial_k g_{nl} + \partial_l g_{kn} - \partial_n g_{kl}) \quad (6)$$

Note.

The Beltrami operator of the first kind for one function is by notation

$$\Delta_1 \Phi := \frac{1}{2} \Delta_1(\Phi, \Phi) = \sum_{i,j=1}^N (g_{ij})^t \frac{\partial \Phi}{\partial u_i} \frac{\partial \Phi}{\partial u_j} = \sum_{i,j=1}^N (g_{ij})^t \partial_i \Phi \partial_j \Phi \quad (7)$$

2 Propositions

Proposition 1.

$$\Delta_1(\Phi \Psi) = \Phi^2 (\Delta_1 \Psi) + (\Delta_1 \Phi) \Psi^2 + \Phi \Psi \Delta_1(\Phi, \Psi) \quad (8)$$

Proof.

The result follows from the differentiation of the product of two functions and the fact that g_{ij} is symmetric (we drop the notation $(g_{ij})^t$ using the g_{ij} as $(g_{ij})^t$):

$$\begin{aligned}\Delta_1(\Phi\Psi) &= \sum_{i,j=1}^N g_{ij} \partial_i(\Phi\Psi) \partial_j(\Phi\Psi) = \sum_{i,j=1}^N g_{ij} [(\Psi(\partial_i\Phi) + \Phi(\partial_i\Psi))((\partial_j\Phi)\Psi + \Phi(\partial_j\Psi))] = \\ &= \sum_{i,j=1}^N g_{ij} [\Psi^2(\partial_i\Phi)(\partial_j\Phi) + \Phi^2(\partial_i\Phi)(\partial_j\Psi) + \Phi\Psi(\partial_i\Phi)(\partial_j\Psi) + \Phi\Psi(\partial_j\Phi)(\partial_i\Psi)] = \\ &= \Phi^2(\Delta_1\Psi) + \Psi^2(\Delta_1\Phi) + \Phi\Psi\Delta_1(\Phi, \Psi)\end{aligned}$$

Proposition 2.

$$\Delta_1(\Phi\Psi, Z) = \Phi\Delta_1(\Psi, Z) + \Psi\Delta_1(\Phi, Z) \quad (9)$$

Proof.

$$\begin{aligned}\Delta_1(\Phi\Psi, Z) &= 2 \sum_{i,j=1}^N g_{ij} \partial_i(\Phi\Psi) \partial_j Z = 2 \sum_{i,j=1}^N g_{ij} (\Psi \partial_i \Phi + \Phi \partial_i \Psi) \partial_j Z = \\ &= 2\Phi \sum_{i,j=1}^N g_{ij} (\partial_i \Psi) (\partial_j Z) + 2\Psi \sum_{i,j=1}^N g_{ij} (\partial_i \Phi) (\partial_j Z)\end{aligned}$$

and the result follows from the definition of $\Delta_1(., .)$.

One can observe from Proposition 2 that Beltrami's differential operator for products, obeys the same rule as the classical differential operator $\frac{d}{dx}$ of functions of one variable. For example set $\Phi = \Psi$ in (9), we get:

$$\Delta_1(\Phi^2, Z) = 2\Phi\Delta_1(\Phi, Z) \quad (10)$$

From Propositions (1) and (2) we get by induction

Proposition 3.

If $n = 1, 2, 3, \dots$, then

$$\Delta_1(\Phi^n, \Psi) = n\Phi^{n-1}\Delta_1(\Phi, \Psi) \quad (11)$$

The semilinear property which is easy someone to see is

$$\Delta_1(\Phi + \Psi, Z) = \Delta_1(\Phi, Z) + \Delta_1(\Psi, Z) \quad (12)$$

Also if f is a function such that

$$f(z) = \sum_{n=1}^{\infty} c_n z^n \quad (13)$$

then

Theorem 1.

$$\Delta_1(f(\Phi), Z) = f'(\Phi)\Delta_1(\Phi, Z) \quad (14)$$

Proof.

It follows from Proposition 3 and the semilinear property.

Theorem 1 will help us to evaluate functions of the second Beltrami derivative.

Proposition 4.

$$\Delta_2(\Phi\Psi) = \Psi\Delta_2(\Phi) + \Phi\Delta_2(\Psi) + \Delta_1(\Phi, \Psi) \quad (15)$$

Proof.

From the relations

$$\begin{aligned} \frac{\partial^2(\Phi\Psi)}{\partial x\partial y} &= \frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial x\partial y} + \frac{\partial\Phi}{\partial x} \frac{\partial\Psi}{\partial y} + \frac{\partial\Phi}{\partial y} \frac{\partial\Psi}{\partial x} \\ \frac{\partial(\Phi\Psi)}{\partial x} &= \frac{\partial\Phi}{\partial x} \Psi + \frac{\partial\Psi}{\partial x} \Phi \end{aligned}$$

and the definitions, of 1 and 2-Beltrami derivatives, the proof easily follows.

Proposition 5.

For $n = 2, 3, \dots$

$$\Delta_2(\Phi^n) = n\Phi^{n-1}\Delta_2(\Phi) + n(n-1)\Phi^{n-2}\Delta_1(\Phi) \quad (16)$$

Proof.

Set $\Phi = \Psi$ in (15) then

$$\Delta_2(\Phi^2) = 2\Phi\Delta_2(\Phi) + \Delta_1(\Phi, \Phi) \quad (17)$$

Also

$$\Delta_2(\Phi^3) = \Delta_2(\Phi\Phi^2) = \Phi^2\Delta_2(\Phi) + \Phi\Delta_2(\Phi^2) + \Delta_1(\Phi^2, \Phi)$$

But from Theorem 1. and (17) we have

$$\begin{aligned} \Delta_2(\Phi^3) &= \Phi^2\Delta_2(\Phi) + \Phi(2\Phi\Delta_2(\Phi) + \Delta_1(\Phi, \Phi)) + 2\Phi\Delta_1(\Phi, \Phi) = \\ &= 3\Phi^2\Delta_2(\Phi) + 3\Phi\Delta_1(\Phi, \Phi) \end{aligned}$$

The result as someone can see follows easy from (10), (15) and (17) by induction.

From the linearity of the 2-Beltrami derivative and (16) we get

Theorem 2.

If f is analytic around 0 we have

$$\Delta_2(f(\Phi)) = f'(\Phi)\Delta_2(\Phi) + f''(\Phi)\Delta_1(\Phi) \quad (18)$$

From Theorem 2 we get that if for some Φ in some space holds

$$\Delta_1(\Phi) = \Delta_2(\Phi) = 0 \quad (19)$$

then for every function f analytic in a open set containing the origin we have

$$\Delta_2(f(\Phi)) = 0 \quad (20)$$

Lemma 1.

Set

$$|u| = \sqrt{\sum_{k=1}^N u_k^2}$$

and

$$\|u\|^2 = \sum_{i,j=1}^N (g_{ij})^t u_i u_j$$

then

$$\Delta_1(|u|) = \frac{\|u\|^2}{|u|^2} \quad (21)$$

Proof.

$$\begin{aligned} \Delta_1(|u|) &= \sum_{i,j=1}^N (g_{ij})^t \frac{\partial |u|}{\partial u_i} \frac{\partial |u|}{\partial u_j} = \sum_{i,j=1}^N (g_{ij})^t \frac{\partial \sqrt{\sum_{k=1}^N u_k^2}}{\partial u_i} \frac{\partial \sqrt{\sum_{k=1}^N u_k^2}}{\partial u_j} = \\ &= \frac{1}{4} \sum_{i,j=1}^N (g_{ij})^t 2u_i 2u_j \frac{1}{4|u|^2} = \frac{\|u\|^2}{|u|^2} \end{aligned}$$

One can see also that

$$\Delta_1(\Phi(|u|), \Psi(|u|)) = \Phi'(|u|) \Psi'(|u|) \frac{\|u\|^2}{|u|^2} \quad (22)$$

$$\Delta_2(f(|u|)) = f'(|u|) \Delta_2(|u|) + f''(|u|) \frac{\|u\|^2}{|u|^2} \quad (23)$$

3 The Euclidian Space and the Polar Coordinates

In the Euclidean space \mathbf{E}_3 the metric is $g_{ij} = \delta_{ij}$ and

$$\Delta_2(\Phi) = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

We make the change of coordinates

$$x = r \sin(\theta) \cos(\phi), y = r \sin(\theta) \sin(\phi), z = r \cos(\theta)$$

then

$$\Delta_2(\Phi) = \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \left(\frac{1}{\sin^2(\theta)} \frac{\partial^2 \Phi}{\partial \phi^2} + \cot(\theta) \frac{\partial \Phi}{\partial \theta} + \frac{\partial^2 \Phi}{\partial \theta^2} \right)$$

and holds

$$\Delta_2(f(e^{-i\phi} r \sin(\theta))) = \Delta_1(f(e^{-i\phi} r \sin(\theta))) = 0 \quad (24)$$

Corollary If $r =$

Expanding this idea in \mathbf{E}_N -Euclidian space with dimension $N = 2n$, for any function f analytic at the origin:

Theorem 3.

For f analytic in the origin set

$$\Phi_0(r, \theta_1, \dots, \theta_{N-1}) = r e^{i\theta_1} e^{\psi(\theta_2)} e^{i\theta_3} e^{\psi(\theta_4)} \dots e^{i\theta_{N-2}} e^{\psi(\theta_{N-1})} \quad (25)$$

where $N = 2n$, $n = 1, 2, 3, \dots$ and

$$\psi(x) = 2 \arctan \left(\tan \left(\frac{x}{2} \right) \right) \quad (26)$$

Then

$$\Delta_{1S}(f(\Phi_0)) = 0 \quad (27)$$

Proof.

If $x = (x_1, x_2, \dots, x_N)$ is an element of \mathbf{R}^N then we have

$$r = \sqrt{x_1^2 + x_2^2 + \dots + x_N^2}$$

and

$$\begin{aligned} \theta_1, \theta_2, \dots, \theta_{N-1} &\in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \\ \theta_{N-1} &\in (0, 2\pi) \end{aligned}$$

then

$$\begin{aligned} x_1 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-2}) \cos(\theta_{N-1}), \\ x_2 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-2}) \sin(\theta_{N-1}), \\ x_3 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-3}) \sin(\theta_{N-2}), \\ x_4 &= r \cos(\theta_1) \cos(\theta_2) \dots \cos(\theta_{N-4}) \sin(\theta_{N-3}), \\ &\vdots \\ x_{N-1} &= r \cos(\theta_1) \sin(\theta_2), \\ x_N &= r \sin(\theta_1). \end{aligned} \quad (28)$$

Writing

$$\Phi_1(u_1, u_2, \dots, u_N) = a_1(u_1) a_2(u_2) \dots a_N(u_N)$$

we try to solve the spherical partial differential equation

$$\Delta_{1S}(\Phi_1) = 0$$

(by Δ_{1S} we note the 1-Beltrami operator in spherical coordinates).

One can manage to find g_{ij} at any rate N , using Mathematica Program. We first find the solutions for $N = 2, 3, 4, 5, 6, 7$ and then we derive numerically our results for arbitrary dimensions. Actually the operator $\Delta_{1S}(\Phi_1)$ is quite simple, since the g_{ij} form a diagonal matrix:

$$\begin{aligned} & \left(u_1 \prod_{k=1}^N a_k(u_k) \right)^{-2} \Delta_{1S}(\Phi_1) = \\ & = \frac{a'_1(u_1)^2 u_1^2}{a_1(u_1)^2} + \frac{a'_2(u_2)^2}{a_2(u_2)^2} + \sum_{k=3}^N \frac{a'_k(u_k)^2}{a_k(u_k)^2} \prod_{j=2}^{k-1} \sec(u_j)^2 = 0 \end{aligned} \quad (29)$$

Hence the problem reduces to prove that g_{ij} is diagonal and the values in the diagonal are

$$\left\{ 1, u_1^2, u_1^2 \cos(u_2)^2, u_1^2 \cos(u_2)^2 \cos(u_3)^2, \dots, u_1^2 \prod_{j=2}^{N-1} \cos(u_j)^2 \right\}$$

If we assure that then we solve the three simple differential equations $xy'/y = c$, $y'/y = 1$ and $y'/y = c \sec(x)$ and the result follows.

Another interesting proposition for evaluations of spherical 2-Beltrami operators is

Proposition 6.

Let f be analytic in a open set containing the origin.

i) Let also

$$F(x, y, \dots, z) = f(\Phi_0(x, y, \dots, z))$$

then

$$\Delta_{2S}(F) = f'(\Phi_0) \Delta_{2S}(\Phi_0) \quad (30)$$

ii) Further if $N = 2n$, $n = 2, 3, \dots$ and

$$\begin{aligned} & G(x_1, x_2, x_3, x_4, \dots, x_{N-3}, x_{N-2}, x_{N-1}, x_N) = \\ & = f_1(x_1 x_2) f_2(x_3 x_4) \dots f_{n-1}(x_{N-3} x_{N-2}) f_n(x_{N-1} x_N) \end{aligned}$$

where $x_1 = r$, $x_2 = e^{i\theta_1}$, $x_3 = e^{\psi(\theta_2)}$, \dots , $x_{N-1} = e^{i\theta_{N-2}}$, $x_N = e^{\psi(\theta_{N-1})}$ then also

$$\Delta_{1S}(G) = 0 \quad (31)$$

where f_j , $j = 1, 2, \dots, n$ as f .

Proof.

i) Equation (30) follows immediately from Theorem 2,3.

ii) Equation (31) follows from (30) where we can set the terms of (29) respectively with $p_1, p_2, \dots, p_{N-1}, p_N$ and then take $p_k + p_{k+1} = 0$, for the equation to hold (Note that we use, as in Theorem 3, the method of separate variables). The p 's in the general solution appear as powers $e^{i(p_k \theta_k + p_{k+1} \psi(\theta_{k+1}))}$.

References

- [1]: Pages stored in the Web.
- [2]: Graduate books in Greek.